# A FUNDAMENTAL REMARK CONCERNING DETERMINANTAL NOTATIONS WITH THE EVALUATION OF AN IMPORTANT DETERMINANT OF SPECIAL FORM.*

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## 1. INTRODUCTION.

THE fundamental evaluation-theorem $T_2$ and its generalization $T_s$ to be discussed in this paper (§§2, 3, 6; §5) were first derived by Professor W. H. Metzler as generalizations of a certain theorem $T_2\ (m, n = 2,\ 2)$ of Muir (given in §2).

The remark concerning determinantal notations is made in §3 as preparatory to and illustrated by the convenient formulations of the set of related theorems given in §§3, 4, 5. The theorem $T_s$ comes from $T_2$ by induction (§5). Of $T_2$ I give in § 6 four proofs; these proofs are, in a general way, characterized at the beginning of §6; also in these proofs the notations employed exhibit their flexibility.

The special case $T_2'$ of $T_2$ (§4) I had found in algebraic investigations, and on learning, in November, 1898, of the Muir theorem the generalizations to $T_2$ and $T_s$ were immediately made.

Professor Metzler presented his results to the London Mathematical Society in a paper read Jan. 12, 1899, entitled "On a Determinant each of whose Elements is the Product of $k$ Factors." (This $k$ is the $s$ of $T_s$). He makes use of the ordinary unipartite notation which is not so well fitted to express the essential properties of the special determinants in question as is the multipartite notation here used (§§3, 4, 5).†

The theorems are of importance in algebra, and the determinants will be found occurring in algebra (in real life, one may say) exactly in this multipartite notation.

According to Hensel (*Acta Mathematica*, Vol. 14, p. 319, 1891) the special case $T_2''$ of $T_2'$ (§4) is due to Kronecker. His proof (in lectures on

* This paper was read before the Chicago section of the American Mathematical Society at the meeting of April 14, 1900.

† This paper has just appeared in *The American Mathematical Monthly*. [Note added July 10, 1900.]

algebra) was by "eine elegante Umformung der zu untersuchenden Determinante mit Hülfe des Multiplicationssatzes." Rados has given a proof by means of Grassmann's theory. Hensel has given a proof (*l. c.*, p. 317) of which my proof III (§6) of $T_2$ is a modified generalization

## 2. Preliminary Statement of the Fundamental Theorem $T_2$. Special Cases.

Theorem $T_2$. *The product of n determinants each of order m and m determinants each of order n may be expressed as a determinant of order mn each of whose constituent elements is the product of two factors.*

Examples. We take the elements of each determinant in the double suffix notation, using the skeleton letters $b'$ $b''$ . . for the elements of the determinants of order $m$ and $c'$ $c''$ . . for those of order $n$, and the corresponding letters $B'$ $B''$ . . $C'$ $C''$ . . for the determinants themselves.

*The case* $(m, n) = (2, 2)$.

$$B'\,B''\,C'\,C''=$$

$$\begin{vmatrix} b'_{11} & b'_{12} & 0 & 0 \\ b'_{21} & b'_{22} & 0 & 0 \\ 0 & 0 & b''_{11} & b''_{12} \\ 0 & 0 & b''_{21} & b''_{22} \end{vmatrix} \cdot \begin{vmatrix} c'_{11} & 0 & c'_{12} & 0 \\ 0 & c''_{11} & 0 & c''_{12} \\ c'_{21} & 0 & c'_{22} & 0 \\ 0 & c''_{21} & 0 & c''_{22} \end{vmatrix} = \begin{vmatrix} b'_{11}c'_{11} & b'_{12}c''_{11} & b'_{11}c'_{12} & b'_{12}c''_{12} \\ b'_{21}c'_{11} & b'_{22}c''_{11} & b'_{21}c'_{12} & b'_{22}c''_{12} \\ b''_{11}c'_{21} & b''_{12}c''_{21} & b''_{11}c'_{22} & b''_{12}c''_{22} \\ b''_{21}c'_{21} & b''_{22}c''_{21} & b''_{21}c'_{22} & b''_{22}c''_{22} \end{vmatrix}.$$

This example, in less suggestive notation, is given by Muir: *Theory of Determinants*, p. 117, 1882. In the multiplication of the two determinants of order four one takes the $b$-rows with the $c$-columns.

*The case* $(m, n) = (3, 2)$.

$$B'\,B''\,C'\,C''\,C'''=$$

$$\begin{vmatrix} b'_{11} & b'_{12} & b'_{13} & 0 & 0 & 0 \\ b'_{21} & b'_{22} & b'_{23} & 0 & 0 & 0 \\ b'_{31} & b'_{32} & b'_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & b''_{11} & b''_{12} & b''_{13} \\ 0 & 0 & 0 & b''_{21} & b''_{22} & b''_{23} \\ 0 & 0 & 0 & b''_{31} & b''_{32} & b''_{33} \end{vmatrix} \cdot \begin{vmatrix} c'_{11} & 0 & 0 & c'_{12} & 0 & 0 \\ 0 & c''_{11} & 0 & 0 & c''_{12} & 0 \\ 0 & 0 & c'''_{11} & 0 & 0 & c'''_{12} \\ c'_{21} & 0 & 0 & c'_{22} & 0 & 0 \\ 0 & c''_{21} & 0 & 0 & c''_{22} & 0 \\ 0 & 0 & c'''_{21} & 0 & 0 & c'''_{22} \end{vmatrix} = \begin{vmatrix} b'_{11}c'_{11} & b'_{12}c''_{11} & b'_{13}c'''_{11} & b'_{11}c'_{12} & b'_{12}c''_{12} & b'_{13}c'''_{12} \\ b'_{21}c'_{11} & b'_{22}c''_{11} & b'_{23}c'''_{11} & b'_{21}c'_{12} & b'_{22}c''_{12} & b'_{23}c'''_{12} \\ b'_{31}c'_{11} & b'_{32}c''_{11} & b'_{33}c'''_{11} & b'_{31}c'_{12} & b'_{32}c''_{12} & b'_{33}c'''_{12} \\ b''_{11}c'_{21} & b''_{12}c''_{21} & b''_{13}c'''_{21} & b''_{11}c'_{22} & b''_{12}c''_{22} & b''_{13}c'''_{22} \\ b''_{21}c'_{21} & b''_{22}c''_{21} & b''_{23}c'''_{21} & b''_{21}c'_{22} & b''_{22}c''_{22} & b''_{23}c'''_{22} \\ b''_{31}c'_{21} & b''_{32}c''_{21} & b''_{33}c'''_{21} & b''_{31}c'_{22} & b''_{32}c''_{22} & b''_{33}c'''_{22} \end{vmatrix}.$$

The reader is requested to write down the corresponding examples of the theorem for the cases $(m, n) = (3, 3), (4, 2), (4, 3), (4, 4)$.

## 3. Definitive Statement of the Fundamental Theorem $T_2$: An Illustration of a Fundamental Remark concerning Determinantal Notations.

We use for the general case $(m, n)$ the notations of §2 with the further permanent understanding that the indices $f, g, h$ shall have the values $1, \ldots, m$ and the indices $i, j, k$ the values $1, \ldots, n$. We have then

$$B^{(\ )} = |b_{fh}^{(\ )}|, \quad C^{(\ )} = |c_{ik}^{(\ )}|$$

A scrutiny of the examples of §3 reveals the fact that for the rather obvious generalization of those examples the elements $a_{uv}$ $(u, v = 1, 2, \ldots, mn)$ of the product determinant $A$ of order $mn$ have the general form

$$b_{fh}^{(i)}\, c_{ik}^{(h)}.$$

There is a definite interdependence of the $u\,v$ and the $f\,h\,i\,k$. This interdependence is sufficiently simple; its explicit expression, however, leads to cumbrous forms.

It is now in place to make a *remark of fundamental import with respect to determinants* which has received, if any, insufficient notice in the text-books. *A determinant of order $t$ is uniquely defined by the unique definition of its $t^2$ elements in the form $a_{uv}$ where the suffixes $u\,v$ run independently over any (the same) set of $t$ distinct marks* (OF ANY DESCRIPTION WHATEVER).

The point is this: one ordinarily uses the set of marks $1, 2, \ldots, t$ not only to define the elements $a_{uv}$ but further to localize them as the elements of the familiar square array of $t$ rows and $t$ columns. Now the determinant $A$ *is* a certain rational integral function of its $t^2$ elements $a_{uv}$. For the investigation of the properties of general determinants this function *is conveniently denoted* by this square array in the determinant brackets. But for the investigation of determinants of special forms (as in the case now in hand) it is often convenient to introduce some other set of $t$ marks.

The truth of the remark follows from the fact that *however* the notations $\mu_1, \mu_2, \ldots, \mu_t$, be assigned to the $t$ marks, the determinant $|a_{\mu_r \mu_s}|$ of the square array

$$\begin{matrix} a_{\mu_1 \mu_1} & \cdot & \cdot & a_{\mu_1 \mu_t} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{\mu_t \mu_1} & \cdot & \cdot & a_{\mu_t \mu_t} \end{matrix}$$

is always the same, the various arrays coming from one of them by various cogredient permutations of its rows and columns. This definite function of the $t^2$ elements $a_{uv}$ is the determinant $|a_{uv}|$; it is uniquely defined by its notation with the understanding that the suffixes $u, v$ run independently over a certain set of $t$ distinct marks. Thus the determinant

$$|a_{uv}| \qquad (u, v = \kappa, \lambda, \mu)$$

is the function

$$a_{\kappa\kappa}\,a_{\lambda\lambda}\,a_{\mu\mu} + a_{\kappa\lambda}\,a_{\lambda\mu}\,a_{\mu\kappa} + a_{\kappa\mu}\,a_{\lambda\kappa}\,a_{\mu\lambda} - a_{\kappa\kappa}\,a_{\lambda\mu}\,a_{\mu\lambda} - a_{\kappa\mu}\,a_{\lambda\lambda}\,a_{\mu\kappa} - a_{\kappa\lambda}\,a_{\lambda\kappa}\,a_{\mu\mu}.$$

Also in these determinants $|a_{uv}|$ it is convenient to speak of the row $u_0$ of the $t$ elements $a_{u_0 v}$, and of the column $v_0$ of the $t$ elements $a_{uv_0}$ and of the principal diagonal the product of the $t$ elements $a_{uu}$, (where in each instance one subscript index is indeterminate), although in general there is no first, or second, or $i$th row or column.

Of course the $t$ marks may be themselves multipartite symbols. Indeed for the theorem $T_2$ we have $t = mn$, and find it altogether desirable to use the set of $t$ bipartite marks

$$gj \qquad \begin{pmatrix} g = 1, \ldots, m \\ j = 1, \ldots, n \end{pmatrix}.$$

Thus the element $a_{uv}$ shall be denoted by

$$a_{fi\,hk}.$$

In order to find the precise mode in which each element $a_{fi\,hk}$ depends upon its suffix, we recur to the case $(m, n) = (3, 2)$; using for the six rows and the six columns the bipartite notation $gj$ ($g = 1, 2, 3$; $j = 1, 2$) in the order 11, 21, 31, 12, 22, 32 we find

$$a_{fi\,hk} = b^{(i)}_{fh} \cdot c^{(h)}_{ik}.$$

We now state the

Fundamental Theorem $T_2$. *The determinant*

(1) $$A = |a_{fi\,hk}|$$

*of order $mn$, where throughout*

(2) $$a_{fi\,hk} = b^{(i)}_{fh} \cdot c^{(h)}_{ik} \qquad \begin{pmatrix} f, h = 1, \ldots, m \\ i, k = 1, \ldots, n \end{pmatrix},$$

*is the product of the $n$ determinants $B^{(i)}$ of order $m$ and the $m$ determinants $C^{(h)}$ of order $n$:*

(3) $$B^{(i)} = |b^{(i)}_{fh}|, \quad C^{(h)} = |c^{(h)}_{ik}| \qquad \begin{pmatrix} f, h = 1, \ldots, m \\ i, k = 1, \ldots, n \end{pmatrix};$$

(4) $$A = B^{(1)} \ldots B^{(n)} \cdot C^{(1)} \ldots C^{(m)}.$$

## 4. Corollaries to the Fundamental Theorem $T_2$.

One has, corresponding to the two specializations

$$(',") \qquad a_{fihk} = b_{fh}\ c_{ik}^{(h)}, \quad b_{fh}\ c_{ik} \qquad \begin{pmatrix} f, h = 1, \ldots, m \\ i, k = 1, \ldots, n \end{pmatrix},$$

of the fundamental relation (2), the two theorems:

$$(T_2', T_2'') \quad |a_{fihk}| = |b_{fh}|^n \cdot \prod_{h=1,m} |c_{ik}^{(h)}| \quad , \quad |b_{fh}|^n \cdot |c_{ik}|^m \begin{pmatrix} f, h = 1, \ldots, m \\ i, k = 1, \ldots, n \end{pmatrix}.$$

## 5. The General Theorem $T_s$.

The fundamental theorem $T_2$ dealing with determinants of the *two* orders $m\,n$ I restate in a form suggesting its generalization:

Fundamental Theorem ($T_2$). *The determinant*

$$(1') \qquad A = |a_{g_1 g_2 k_1 k_2}| \qquad \begin{pmatrix} g_1, k_1 = 1, \ldots, n_1 \\ g_2, k_2 = 1, \ldots, n_2 \end{pmatrix}$$

*of order $n = n_1 n_2$, where throughout*

$$(2') \qquad a_{g_1 g_2 k_1 k_2} = a_{g_1 k_1}^{(1 g_2)} \cdot a_{g_2 k_2}^{(2 k_1)},$$

*is the product of the $\nu_1 = n/n_1 = n_2$ determinants $A^{(1 g_2)}$ of order $n_1$ and the $\nu_2 = n/n_2 = n_1$ determinants $A^{(2 k_1)}$ of order $n_2$:*

$$(3') \qquad A^{(1 g_2)} = |a_{g_1 k_1}^{(1 g_2)}|, \quad A^{(2 k_1)} = |a_{g_2 k_2}^{(2 k_1)}| \qquad \begin{pmatrix} g_1, k_1 = 1, \ldots, n_1 \\ g_2, k_2 = 1, \ldots\ n_2 \end{pmatrix}$$

General Theorem ($T_s$). — *The determinant*

$$(1'') \qquad A = |a_{g_1 \ldots g_s k_1 \ldots k_s}| \qquad \begin{pmatrix} g_\beta, k_\beta = 1, \ldots, n_\beta \\ \beta = 1, \ldots, s \end{pmatrix}$$

*of order $n = n_1 n_2 \ldots n_s$, where throughout*

$$(2'') \qquad a_{g_1 \ldots g_s k_1 \ldots k_s} = \prod_{\beta = 1, s} a_{g_\beta k_\beta}^{(\beta g_{\beta+1} \ldots k_{\beta-1})},$$

*(the superscript $(\beta g_{\beta+1} \ldots k_{\beta-1})$ being, aside from the $\beta$, the $s-1$ indices lying between $g_\beta$ and $k_\beta$ on the face of the double $s$-partite suffix $g_1 \ldots g_s k_1 \ldots k_s$), is the product of certain $\nu = \nu_1 + \ldots + \nu_s$ determinants, viz. for $\beta = 1, 2, \ldots, s$ the $\nu_\beta = n/n_\beta$ determinants*

$$(3'') \qquad A^{(\beta g_\beta + 1 \ldots k_{\beta-1})} = \left| a_{g_\beta k_\beta}^{(\beta g_\beta + 1 \ldots k_{\beta-1})} \right| \qquad (g_\beta, k_\beta = 1, \ldots, n_\beta)$$

*of order $n_\beta$.*

This theorem $T_s$ is easily proved by induction. The fundamental theorem $T_2$ is taken as proved (*cf.* §6). We assume the truth of $T_{s-1}$. Writing for $k_1 = 1, \ldots, n_1$

$$A^{(k_1)} = | a^{(k_1)}_{\gamma\kappa} | = | a^{(k_1)}_{g_2 \ldots g_s k_2 \ldots k_s} | \qquad \begin{pmatrix} \gamma, \kappa = 1, \ldots, \nu_1 \\ g_\beta, k_\beta, = 1 \ldots, n_\beta \\ (\beta = 2, \ldots, s) \end{pmatrix}$$

where throughout (for a certain $1-1$ correspondence between the $\nu_1$ marks $1, \ldots, \nu_1$ and the $\nu_1$ $\overline{s-1}$-partite marks $f_2 \ldots f_s$ ($f_\beta = 1, \ldots, n_\beta$; $\beta = 2, \ldots, s$) in which $\gamma$ and $g_2 \ldots g_s$ correspond and $\kappa$ and $k_2 \ldots k_s$ correspond)

$$a^{(k_2)}_{\gamma\kappa} = a^{(k_1)}_{g_2 \ldots g_s\, k_2 \ldots k_s} = \prod_{\beta = 2, s} a^{(\beta\, g_{\beta+1} \ldots k_{\beta-1}}_{g_\beta\, k_\beta},$$

one sees (by $T_{s-1}$) that $A^{(k_1)}$ is the product of the $(\nu_2 + \ldots + \nu_s)/n_1$ determinants $A^{(\beta\, g_{\beta+1} \ldots k_{\beta-1})}$ ($g_\beta, k_\beta = 1, \ldots, n_\beta$, for $\beta = 2, \ldots, s$), and since throughout

$$a_{g_1 g_2 \ldots g_s k_1 k_2 \ldots k_s} = a_{g_1\gamma\, k_1\kappa} = a^{(1\,\gamma)}_{g_1\, k_1} a^{(k_1)}_{\gamma\kappa}$$

one sees (by $T_2$) that $A$ is the product of the $\nu_1$ determinants $A^{(1\gamma)} = A^{(1\, g_2 \ldots g_s)}$ and the $n_1$ determinants $A^{(k_1)}$. Hence the truth of $T_s$ is seen to follow from that of $T_2$ and $T_{s-1}$.

## 6. Four Proofs of the Fundamental Theorem $T_2$.

The four proofs I — IV depend essentially upon the following theorems in the theory of determinants:

Proof I: upon the *multiplication theorem*, so that this proof is a generalization of Muir's proof for the case $(m,n) = (2,2)$.

Proof II: upon the *Laplacian development* of a determinant.

Proof III: upon the theorem that *the vanishing of the determinant of a system of $t$ linear homogeneous equations in $t$ unknowns is the necessary and sufficient condition for the existence of a non-zero solution of the system of equations.*

Proof IV: upon *three fundamental explicit formulas for the determinant* $| a_{uv} |$, by means of which one makes an explicit transformation of the right side of the identity (4) into its left side.

Proofs II, III: further on the theorems *concerning the decomposition of rational integral functions of one or several indeterminates into irreducible functions of the same kind* (*e. g.*, Weber's *Algebra*, Vol. 1, §51), and on the fact that *the general determinant is irreducible.*

**Proof 1.** Recalling the statement at the close of §3 of the fundamental theorem $T_2$, we take two determinants $B$, $C$ of order $mn$, in the bipartite notation used for $A$ in (1). The determinant $A$ is the row-by-column product of the determinants $B$ and $C$, if throughout

$$a_{fi\,hk} = \sum_{g,j} b_{fi\,gj}\, c_{gj\,hk}. \tag{5}$$

This relation (5) is identical with the known relation (2), and so will follow

$$A = B\cdot C, \tag{6}$$

if (1°) $b_{fi\,gj}$ is 0 unless $j = i$, (2°) $c_{gj\,hk}$ is 0 unless $g = h$, (so that the right of (5) reduces to $b_{fi\,hi}\, c_{hi\,hk}$), (3°) $b_{fi\,hi}$ is $b^{(i)}_{fh}$, and (4°) $c_{hi\,hk}$ is $c^{(h)}_{ik}$.

We suppose that the elements of the determinants $B, C$ are those specified by these conditions (1°) . . (4°). Then $A = B\,C$.

Now, if the $mn$ rows and columns of $B$ be arranged in sets of $n$, the second index of the bipartite notation indicating the set and the first index the row or column of the set (as at the close of §3 for the case $(m, n) = (3, 2)$), we see that the determinant $B$ of order $mn$ is the product of the $n$ determinants $B^{(i)}$ of order $m$. Again, if the $mn$ rows and columns of $C$ be arranged in sets of $m$, the first index indicating the set and the second index the row or column of the set, we see that the determinant $C$ is the product of the $m$ determinants $C^{(h)}$. Hence the theorem is proved:

$$A = B^{(1)}\,.\,.\,B^{(n)}\,.\,C^{(1)}\,.\,.\,C^{(m)}. \tag{4}$$

**Proof II.** We separate the $mn$ rows $fi$ of $A$ into $n$ sets of $m$ each, the $m$ of each set having the same second index $i$ and distinct first indices $f$. In the set of $m$ rows $fi$ $(i = i_0;\ f = 1, 2, \ldots, m)$ the column $hk$ has the elements

$$a_{fi_0\,hk} = b^{(i_0)}_{fh}\cdot c^{(h)}_{i_0k} \qquad (f = 1, \ldots, m),$$

with the common factor $c^{(h)}_{i_0k}$. We take any $m$ distinct columns $h_1k_1, \ldots, h_mk_m$ of these $m$ rows and obtain a minor of order $m$ for use in the Laplacian development of the determinant $A$; this minor is, apart from sign,

$$\left|a_{fi_0\,h_gk_g}\right|_{f,g=1,\ldots,m} = \left|b^{(i_0)}_{fh_g}\, c^{(h_g)}_{i_0k_g}\right|_{f,g=1,\ldots,m} = \prod_{g=1,m} c^{(h_g)}_{i_0k_g}\cdot\left|\ b^{(i_0)}_{fh_g}\ \right|_{f,g=1,\ldots,m}.$$

Now, if two of the $m$ $h_g$'s are equal, the determinant $|b^{(i_0)}_{fh_g}|$ has two equal columns and vanishes, while, if the $h_1, \ldots, h_m$ are the distinct numbers $1, \ldots, m$ in some order, this determinant is $\pm B^{(i_0)}$. Thus every minor of order $m$ arising from these $m$ rows $fi$ $(i = i_0;\ f = 1, \ldots, m)$ has the factor $B^{(i_0)}$.

By this Laplacian development of $A$ we see that $A$ has the factor

$$B^{(1)} B^{(2)} \ldots B^{(n)} \equiv B \text{ (say)},$$

and by the analogous development, in which are interchanged the rôles of rows and columns on the one hand and of first and second indices on the other hand, we see that $A$ has the factor

$$C^{(1)} C^{(2)} \ldots C^{(m)} \equiv C \text{ (say)}.$$

Now the factors $B^{(i)}$, $C^{(h)}$ being distinct general determinants are $m+n$ distinct irreducible factors of $A$. Hence $A$ has the factor $BC$. $A$ and $BC$ are each homogeneous of degree $2mn$ in the letters $b, c$. Hence $A = \mathbf{a} BC$ where $\mathbf{a}$ is independent of the $b, c$. The literal term

$$\prod_{h,i} b^{(i)}_{hh} c^{(h)}_{ii}$$

occurs in $A$ as the principal diagonal with the coefficient $+1$, and in $BC$ as the product of the principal diagonals of the $B^{(i)}$, $C^{(h)}$ likewise with the coefficient $+1$; it occurs otherwise neither in $A$ nor in $BC$. Hence $\mathbf{a} = 1$, and $A = BC$: the theorem is proved.

**Proof III.** We build the system of $mn$ equations in the $mn$ unknowns $x_{hk}$:

$$(7) \qquad X_{fi} \equiv \sum_{h,k} a_{fi\,hk}\, x_{hk} \equiv \sum_{h,k} b^{(i)}_{fh}\, c^{(h)}_{ik}\, x_{hk} = 0 \qquad \begin{pmatrix} f=1, \ldots, m \\ i=1, \ldots, n \end{pmatrix},$$

where $\mathrm{X}_{fi}$ is a notation for the linear form in the $x_{hk}$. The determinant $A$ of this system vanishes if the system has a non-zero solution. We prove that there is such a solution if any $B^{(i)}$ or $C^{(h)}$ vanishes. From this and the irreducibility of the $B^{(i)}$, $C^{(h)}$ it follows that $A$ has the factors $B^{(i)}$, $C^{(h)}$: whence follows the theorem itself as in the second proof.

Introducing the forms

$$(8) \qquad Y_{hi} \equiv \sum_{k=1,n} c^{(h)}_{ik}\, x_{hk} \qquad \begin{pmatrix} h=1, \ldots, m \\ i=1, \ldots, n \end{pmatrix}$$

we express the system (7) in this way:

$$(9) \qquad \mathrm{X}_{fi} \equiv \sum_{h=1,m} b^{(i)}_{fh}\, Y_{hi} = 0 \qquad \begin{pmatrix} f=1, \ldots, m \\ i=1, \ldots, n \end{pmatrix}.$$

One observes that the unknown $x_{hk}$ enters only the $n$ forms $Y_{hi}$ $(i=1, \ldots, n)$, these $n$ linear forms $Y_{hi}$ involve precisely the $n$ unknowns $x_{hk}$ $(k=1, \ldots, n)$,

and the determinant of this system $(8_h)$ is $C^{(h)}$. Further the form $Y_{hi}$ enters only the $m$ forms $X_{fi}$ $(f=1, \ldots, m)$, these $m$ linear forms $X_{fi}$ involve precisely the $n$ forms $Y_{hi}$ $(h=1, \ldots, m)$, and the determinant of this system $(9_i)$ is $B^{(i)}$.

Now, if one of the $C^{(h)}$, say $C^{(h_0)}$, vanishes, we have a non-zero solution $(x_{h_01}, \ldots, x_{h_0n}) \neq (0, \ldots, 0)$ of the system $(8_{h_0})$ : $Y_{h_0i}=0$ $(i=1, \ldots, n)$, and we take the zero solutions $(x_{h1}, \ldots, x_{hn})=(0, \ldots, 0)$ of the $m-1$ systems $(8_h)$ $(h \neq h_0)$ : $Y_{hi}=0$ $(i=1, \ldots, n)$. Thus we obtain a non-zero solution $(\ldots x_{hk} \ldots)$ of the system $(7)$ : $X_{fi}=0$, by the mediation of the zero solution $(\ldots Y_{hi} \ldots)$ of the system $(9)$.

If, however, no $C^{(h)}$ vanishes while say $B^{(i_0)}$ vanishes, we have a non-zero solution $(Y_{1i_0}, \ldots, Y_{mi_0}) = (y_{1i_0}, \ldots, y_{mi_0})$ of the system $(9_{i_0})$ : $X_{fi_0}=0$ $(f=1, \ldots, m)$, and we take the zero solutions $(Y_{1i}, \ldots, Y_{mi}) = (y_{1i}, \ldots, y_{mi})$ $= (0, \ldots, 0)$ of the remaining $n-1$ systems $(9_i)$ $(i \neq i_0)$ ; thus we obtain a non-zero solution $(\ldots Y_{hi} \ldots) = (\ldots y_{hi} \ldots)$ of the system $(9)$. The $m$ systems $(7_h)$ have non-zero determinants $C^{(h)}$ $(h=1, \ldots, m)$ ; using this non-zero set $(\ldots y_{hi} \ldots)$, we find a solution $(\ldots x_{hk} \ldots)$ of the complete non-homogeneous system:

$$(10) \qquad Y_{hi}=y_{hi} \qquad \begin{pmatrix} h=1, \ldots, m \\ i=1, \ldots, n \end{pmatrix}.$$

This is necessarily a non-zero solution $(\ldots x_{hk} \ldots)$ of the system $(10)$ and as well of the system $(7)$. The third proof of the theorem is now complete.

**Proof IV.** The fourth proof is based on *the fundamental explicit determinantal formulas*:

$$(F_1) \qquad |a_{uv}| = \frac{1}{t!} \sum_{\substack{u_1, \ldots, u_t \\ v_1, \ldots, v_t}} \pm_{u_1 u_2 \ldots u_t} \pm_{v_1 v_2 \ldots v_t} a_{u_1 v_1} a_{u_2 v_2} \cdots a_{u_t v_t},$$

$$(F_2) \qquad \pm_{u_1 \ldots u_t} |a_{uv}| = \sum_{v_1, \ldots, v_t} \pm_{v_1 \ldots v_t} a_{u_1 v_1} \cdots a_{u_t v_t},$$

$$(F_3) \qquad \pm_{v_1 \ldots v_t} |a_{uv}| = \sum_{u_1, \ldots, u_t} \pm_{u_1 \ldots u_t} a_{u_1 v_1} \cdots a_{u_t v_t},$$

where (1°) the determinant $|a_{uv}|$ is one of order $t$ with $t^2$ elements $a_{uv}$ given in the general double-suffix notation, $u\,v$ ranging independently over any (the same) set of $t$ distinct marks, $\mu_1, \ldots, \mu_t$; (2°) in the summations and in the equations the $u_1, \ldots, u_t, v_1, \ldots, v_t$ range independently over this set of $t$ marks; and (3°) the sign-symbol $\pm_{w_1 \ldots w_t}$ (where the $t$ suffixes $w_1 \ldots w_t$ are certain

$t$ marks of this set) has this definition: if two of the $t$ suffixes $w_1 \,..\, w_t$ are equal, then $\pm_{w_1 \,..\, w_t}$ is $0$; if the $t$ suffixes $w_1 \,..\, w_t$ are the $t$ distinct marks $\mu_1, \,..,\, \mu_t$ in some order, then $\pm_{w_1 \,..\, w_t}$ is $+1$ or $-1$ according as of the $\frac{1}{2}t(t-1)$ pairs of suffixes $w_y\, w_z$ ($y, z = 1, \,..,\, t$; $y < z$) the number of pairs, whose elements $w_y\, w_z$ occur in $w_1 \,..\, w_t$ in an order the inverse of that in which they occur in a chosen order-of-reference of the $t$ marks, for example, $\mu_1 \,..\, \mu_t$, is even or odd, where further this order-of-reference may be varied at will from equation to equation. In the general notation every explicit equation will involve terms all of even degree or all of odd degree in these sign-symbols. The product of two sign-symbols is independent of the order-of-reference.

By the use of the general formulas ($F_1, F_2, F_3$) we are to effect a direct explicit transformation of the product

$$B \cdot C \equiv B^{(1)} \,..\, B^{(n)} \cdot C^{(1)} \,..\, C^{(m)} \tag{11}$$

into $A$.

The indices $g\,h\,q$ have the range $1, \,..,\, m$; the indices $i\,j\,r$ have the range $1, \,..,\, n$; the index $p$ has the range $1, \,...,\, l$ where $l = mn$. Between the $l$ distinct marks $p$ and the $l$ distinct marks $qr$ we establish a (any, for the present permanent) $1-1$ correspondence, in which to $l$ corresponds $q_l\, r_l$ and to $qr$ corresponds $p_{qr}$.

For the expression of the determinants $B^{(i)} = | b^{(i)}_{gh} |$ we use formula $F_3$, introducing for every $i$ any particular set of $m$ distinct indices $h_{1i} \,...\, h_{mi}$:

$$B^{(i)} = \sum_{g_{1i}, \,..,\, g_{mi}} \pm_{g_{1i} \,..\, g_{mi}} \pm_{h_{1i} \,..\, h_{mi}} \Pi\, b^{(i)}_{g_{qi}\, h_{qi}}. \tag{12}$$

The order-of-reference is taken to be the natural order $1, \,..,\, m$. The summation-indices $g_{1i} \,..\, g_{mi}$ and the indices $h_{1i} \,..\, h_{mi}$ (which except as to distinctness are at our disposal) are introduced in a notation convenient for the (necessary) transformations of the sequel.

Multiplying together the summations (12) we have by the distributive law:

$$B = \prod_r B = \sum_{\substack{..\, g_{qr} \,.. \\ \binom{q=1,\, m}{r=1,\, n}}} \left[ \left( \prod_r \pm_{g_{1r} \,..\, g_{mr}} \pm_{h_{1r} \,..\, h_{mr}} \right) \left( \prod_{q,\, r} b^{(r)}_{g_{qr}\, h_{qr}} \right) \right]. \tag{13}$$

By means of the correspondence between the $qr$ and the $p$ we introduce throughout notations in $p$ for the notations in $q, r$. Thus, the $l$ $g_{qr}$ are the

$l\, g_p$, the $l\, h_{qr}$ are the $l\, h_p$, the $\prod\limits_{q,r}$ is the $\prod\limits_{l}$ under which the $l\, r$ are the $l\, r_p$. It requires closer consideration to see that

$$(14) \qquad \prod_{r} \pm_{g_{1r}\,..\,g_{mr}} \pm_{h_{1r}\,..\,h_{mr}} = \pm_{g_1r_1 \ldots g_lr_l} \pm_{h_1r_1 \ldots h_lr_l},$$

where the sign-symbols on the right have each $l$ bipartite suffixes $g_pr_p$, $h_pr_p$ and relate to an (*any*) order-of-reference of the $l$ marks $qr$. It is evident, however, that the right side of the equality (14) is only in notation and not in value dependent upon this particular order-of-reference of the $l$ marks $qr$ and upon the fixed correspondence between the $l$ marks $qr$ and the $l$ marks $p$. And in view of the fact that the sign-symbols of the left side of (14) relate, for every value of $r$, to the natural order of the $m$ marks $1, \ldots, m$, one sees immediately the truth of (14) for the following order-of-reference:

$$11, \ldots, q\,1, \ldots, m1,\ 12, \ldots, q\,2, \ldots, m\,2, \ \ldots\ldots,\ 1\,r, \ldots, qr, \ldots, mr, \ \ldots\ldots, \ 1\,n, \ldots, qn, \ldots, mn,$$

and the correspondence relating this order to the natural order of the $l$ marks $p$.

We have then

$$(15) \qquad B = \sum_{g_1, \ldots, g_l} \pm_{g_1r_1 \ldots g_lr_l} \pm_{h_1r_1 \ldots h_lr_l} \prod_p b^{(r_p)}_{g_p h_p},$$

where the notation $r_p$ $(p = 1, \ldots, l)$ relates to the correspondence between the $qr$ and the $p$, while the notation $h_p$ $(p = 1, \ldots, l)$ relates to a permutation $h_p\, r_p$ $(p = 1, \ldots, l)$ of the $l$ $qr$ which is arbitrary as far as the $h_p$ are concerned (for the bipartite marks $h_{qr}\, r$ $(q = 1, \ldots, m;\ r = 1, \ldots, n)$ formed exactly such a permutation). This partially arbitrary permutation $h_p\, r_p$ it is now convenient to fix as the permutation $q_p\, r_p$, so that now

$$(16) \qquad B = \sum_{g_1, \ldots, g_l} \pm_{g_1r_1 \ldots g_lr_l} \pm_{q_1r_1 \ldots q_lr_l} \prod_p b^{(r_p)}_{g_p\, q_p}.$$

By analogous reasoning one has the relation:

$$(17) \qquad C = \sum_{j_1, \ldots, j_l} \pm_{q_1r_1 \ldots q_lr_l} \pm_{q_1j_1 \ldots q_lj_l} \prod_p c^{(q_p)}_{r_p j_p}.$$

Further, on the understanding that all the sign-symbols relate to the same order-of-reference, one has by multiplication the relation:

$$(18) \qquad B\,C = \sum_{\substack{g_1, \ldots, g_l \\ j_1, \ldots, j_l}} \pm_{g_1r_1 \ldots g_lr_l} \pm_{q_1j_1 \ldots q_lj_l} \prod_p b^{(r_p)}_{g_p q_p}\ c^{(q_p)}_{r_p j_p},$$

since $\pm_{q_1r_1 \,.\,.\, q_lr_l}{}^2$ is $+1$. Here the $q_pr_p$ $(p=1, \,.\,., l)$ form any permutation of the $l$ $qr$. We have then an expression for $l!BC$ if the summation on the right of (18) is made also with respect to all values of $q_1, \,.\,., q_l, r_1, \,.\,., r_l$ subject to the restriction that $q_{p'}r_{p'} \neq q_{p''}r_{p''}$ for every $p', p''$ $(p' \neq p'')$. This restriction may, however, be omitted: the summation (18) vanishes if the $q_pr_p$ have a repetition, $q_{p'}r_{p'} = q_{p''}r_{p''}$ $(p' \neq p'')$. To see this we sum first with respect to $g_1, \,.\,., g_l, j_1, \,.\,., j_l$ excepting $g_{p'}g_{p''}$, and then with respect to $g_{p'}, g_{p''}$; for every case $(g_{p'}, g_{p''})$ with $g_{p'} = g_{p''}$ the inner sum vanishes term by term since $\pm_{g_1r_1 \ldots g_lr_l}$ has two equal bipartite suffixes $g_{p'}r_{p'}$, $g_{p''}r_{p''}$; for the two cases $(g_{p'}, g_{p''}) = (g', g'')$, $(g'', g')$ with $g' \neq g''$ the two inner sums cancel each other term by term, the corresponding terms agreeing except as to the signs $\pm_{g_1r_1 \ldots g_lr_l}$ which are either both 0 or one $+1$ and the other $-1$: and so indeed for every $q_pr_p$ $(p=1, \,.\,., l)$ with repetitions the summation (18) vanishes. We may then write

$$(19) \qquad BC = \frac{1}{l!} \sum_{\substack{\cdot\,\cdot\, g_p,\, r_p,\, q_p,\, j_p \,\cdot\,\cdot \\ (p=1, \ldots, l)}}^{\cdot} \pm_{g_1r_1 \ldots g_lr_l} \pm_{q_1j_1 \ldots q_lj_l} \prod_p a_{g_pr_pq_pj_p},$$

where the $a_{gr\,qj}$ have been introduced. But the expression on the right is precisely the evaluation $(F_1)$ in bipartite double-suffix notation of the determinant $A = |a_{gr\,qj}|$. Thus the fundamental theorem is again proved: $A = BC$.

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